

Amendments to the Specification:

Please replace the paragraph beginning at page 1, line 3 with the following amended paragraph:

This application relates to United States Patent Application serial number 09/641,589, filed 08/18/2000, now United States Patent 7,236,953, and incorporated by reference.

Please add the following new paragraphs after the paragraph ending at page 12, line 4:

Calculation of implied probability distributions of future prices

As noted above, the probability distribution data can be calculated using the method described in U.S. Patent 7,236,953.

We first define some relevant terms, with reference to figures 6 through 8. We define x as the strike price, $c(x)$ as the theoretical call price function (the price of the call as a function of strike price), $p(x)$ as the theoretical put price function (figure 6), $F(x)$ as the cumulative distribution function (cdf) of the price of the underlying asset at expiration (figure 7); and $f(x)$ as the probability density function (pdf) of the asset price at expiration (figure 8). By definition, $f(x) = F'(x)$ (i.e., the probability density function is the derivative of the cumulative distribution function). We assume the options are exercisable only on the expiration date, i.e., they are "European-style" options. Even allowing for possible early exercise, ("American-style" options) most liquidly traded call options without large dividends can be treated as if there were no possibility of such exercise, since sale of the option is usually a better alternative; therefore, these call options behave similarly to European-style options. The actual call and put prices are established by options market-makers. Such prices implicitly contain information about a market view of the probability distribution of the price of that asset at the expiration date.

In a simple but precise form, this market view can be stated as follows. Suppose that we were given the call price curve $c(x)$ or the put price curve $p(x)$ as a continuous function of the strike price x for all $x > 0$, as shown in figure 6. Then, the second derivative of either the call or the put price curve is the market view of the risk-neutral probability density function (pdf) $f(x)$

of the asset price at the expiration date, as shown in figure 8. The relationship between $c(x)$, $p(x)$, $f(x)$, and $F(x)$ can be succinctly stated as:

$$F(x) = c'(x) + 1 = p'(x); \quad (1a)$$

$$f(x) = c''(x) = p''(x). \quad (1b)$$

In words, the pdf is the second derivative of either the call price function or of the put price function. A simple proof these relationships follows. As in Hull, we may calculate the European call or put price as an expected value in the risk-neutral distribution.

If the actual value of the asset on the expiration date is v , then the value of a call option at strike price x is $\max\{v - x, 0\}$, and the value of a put option is $\max\{x - v, 0\}$. If the actual value is a random variable with pdf $f(v)$, then the expected value of a call option at x at the expiration date is

$$c_T(x) = E_v[\max\{v - x, 0\}] = \int_x^{\infty} (v - x)f(v)dv, \quad (2a)$$

and the expected value of a put option at x at the expiration date is

$$p_T(x) = E_v[\max\{x - v, 0\}] = \int_0^x (x - v)f(v)dv. \quad (2b)$$

The current values $c(x)$ and $p(x)$ may be obtained by discounting $c_T(x)$ and $p_T(x)$ by e^{-rT} , where r is the risk-free interest rate, but for our purposes, forecasting probability distributions at time T , we do no discounting, and henceforth just write $c(x) = c_T(x)$, $p(x) = p_T(x)$.

Parenthetically, from these expressions we observe that

$$p(x) - c(x) = \int_x^{\infty} (x - v)f(v)dv = x - E_v[v] = x - s^* \quad (2c)$$

where $s^* = E_v[v]$ is the expected value of the asset at the expiration date under the risk-neutral distribution. (If there are no dividends, then $s^* = se^{rT}$; if there are dividends, then in general it is necessary to subtract from se^{rT} the value at time T of the dividends.) This well-known relation is called put-call parity; it shows why either price curve carries the same information.

From the above expression for $c(x)$, it follows that its first derivative is

$$c'(x) = - \int_0^\infty f(x) dx = F(x) - 1, \quad (3)$$

where $F(x) = \int_0^\infty f(v) dx$ is the cumulative distribution function (cdf) of the random variable v ,

as shown in figure 7. To prove this, note that $v - x = \int_x^v dx$. Therefore

$$c(x) = \int_x^\infty (v - x) f(v) dv = \int_x^\infty dv \int_x^v dz f(v) = \int_x^\infty dz \int_x^\infty dv f(v) = \int_x^\infty dz (1 - F(z)) \quad (4)$$

where we interchange the variables v, z to integrate over the two-dimensional region $\mathbb{R} = \{(v, z) : x \leq z \leq v\}$. The last expression implies that $c'(x) = -(1 - F(x))$.

From put-call parity, it follows similarly that

$$p'(x) = 1 + c'(x) = F(x). \quad (5)$$

Since the cdf and pdf are related by $F'(x) = f(x)$, these expressions in turn imply that the second derivative of either $c(x)$ or $p(x)$ is the pdf $f(x)$:

$$c''(x) = p''(x) = F'(x) = f(x). \quad (6)$$

The general character of the option price curves $c(x)$ and $p(x)$ is therefore as follows:

- For all x less than the minimum possible value of v (i. e., such that $F(x) = 0$), $c(x) = E_v[v] - x = s^* - x$ and $p(x) = 0$. In other words, $c(x)$ is a straight line of slope -1 starting at $c(0) = E_v[v] = s^*$, while $p(x) = 0$.
- For all x greater than the maximum possible value of v (i.e., such that $F(x) = 1$), $c(x) = 0$ and $p(x) = x - s^*$. In other words, $p(x)$ is a straight line of slope $+1$ and x -intercept s^* , while $c(x) = 0$.
- These two line segments are joined by a continuous convex \cup curve whose slope increases from -1 to 0 for $c(x)$, and from 0 to $+1$ for $p(x)$.

We note that the fact that the mean $E_v[v]$ of the pdf $f(x)$ is s^* , the value in future dollars at time T of the underlying price s (less the value of any dividends), implies that option prices must be constantly adjusted to reflect changes in the underlying price s , even if there is no market activity in the options.

The fact that $s^* = E_v[v]$ also implies that an option price curve can make no prediction about the general direction of the underlying price s . However, the option price curve does predict the shape of the pdf $f(x)$, and in particular its volatility.

The risk-neutral distribution (at a fixed future time T , for a fixed asset) is defined as the price distribution that would hold if market participants were neutral to risk, which they generally are not. However, many asset pricing theories, such as those underlying Black-Scholes option theory and most of the variations found in the Hull book above, allow for the true risk-averse asset price distribution to be obtained from the risk-neutral distribution $f(x)$ just by adjusting the latter by an appropriate risk premium: If there are no dividends, the true distribution is just $f(xe^{(\mu-r)T})$, where $\mu - r$ is the expected annual return rate for the stock in excess of the risk free rate r . We use a variation on this simple format, slightly modified to allow for dividends, though our invention could also work well with a more complicated adjustment. In this format, a value for $\mu - r$ must still be supplied. We use as a default the "consensus estimate" taken from the textbook "Active Portfolio Management" (1995) by Grinold and Kahn. These authors note a long-term average value of the risk premium to be 6% per year, and suggest multiplying this number by the stock's beta to get $\mu - r$. The parameter beta is the slope of the line giving a regression of the stock in question against a market portfolio, often taken as the S&P 500. This is the well-known CAPM estimate for the expected excess return. Whether good or bad, its stature as a consensus estimate makes it suited to our aim of providing a market view, though it is only a default. Our invention, which provides the risk-neutral component of the probabilities, could work with other estimates for the risk-averse adjustment parameter $\mu - r$ and with any explicit scheme for adjusting the risk neutral probability density to the risk-averse probability density. It is worth pointing out that, for shorter time periods-even a month or two-the risk adjustment required is small and generally overwhelmed by fluctuations in the risk-neutral distribution itself.

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Additional details about these calculations are found in U.S. Patent 7,236,953, as noted above.